# THE KAUFFMAN POLYNOMIALS OF 2-BRIDGE KNOTS
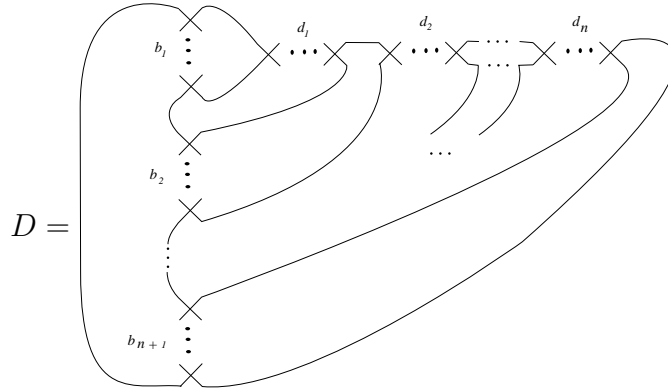
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ABSTRACT. The 2-bridge knots are a family of knots with bridge number 2 [1] [10]. In this paper, we compute the Kauffman polynomials of 2-bridge knots using the Kauffman skein theory and linear algebra techniques. Our calculation can be easily carried out using Mathematica, Maple, Mathcad, etc.

1. INTRODUCTION

The 2-bridge knots (or links) are a family of knots with bridge number 2. A 2-bridge knot (link) has at most 2 components. Except for the knot 8_5, the first 25 knots in the Rolfsen Knot Table are 2-bridge knots. A 2-bridge knot is also called a rational knot because it can be obtained as the numerator or denominator closure of a rational tangle. The rich mathematical aspects of 2-bridge knots can be found in many references such as [3], [4], [7], [10], [6], [12] and [11]. The regular diagram D of a 2-bridge knot can be drawn as follows [10].
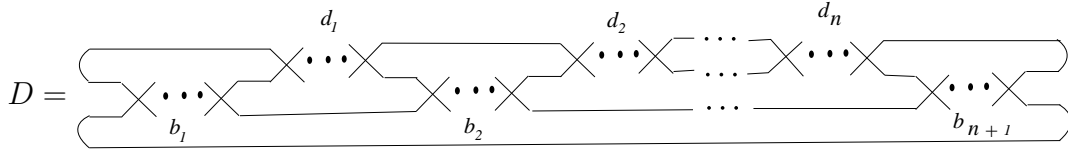
In the diagram, $d_1, d_2, \cdots, d_n, b_1, b_2, \cdots, b_{n+1}$ are nonzero integers whose absolute values indicate the number of crossings. By an isotopy, the diagram D can also be drawn as
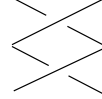
The continued fraction notation for D is $F(D) := [b_1, d_1, b_2, d_2, \cdots, d_n, b_{n+1}]$ [6]. We will work with the diagram above to calculate the Kauffman polynomials of 2-bridge knots. Specific information about twists (crossings) are necessary to identify a 2-bridge knot. Locally, there are four possibilities:
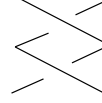

(a) Horizontal left-hand twist


(b) Vertical left-hand twist


(c) Horizontal right-hand twist

(d) Vertical right-hand twist

We choose left-hand twists to be positive twists, and right-hand twists to be negative twists. We will use a positive integer in the regular diagram to indicate the number of crossings in a left-hand twist, and a negative integer to indicate the number of crossings in a right-hand twist. For example, the Whitehead link is a 2-component 2-bridge link with a diagram given by

$$W = $$
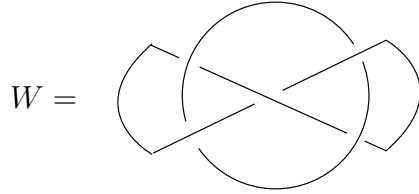
by an isotopy, it can be drawn as

$$W = $$
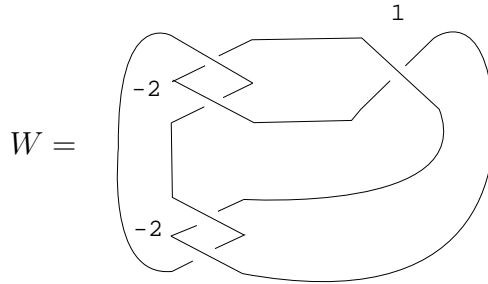
The continued fraction notation for the Whitehead link is $F(W) = [-2, 1, -2]$.

Let $\mathbb{Q}(\alpha, s)$ be the field of rational functions in α, s. By a *framed link* we mean an unoriented link equipped with a nonsingular normal vector field up to homotopy. The links described by figures in this paper will be assigned the vertical framing pointing towards the reader.

There are various versions of *the Kauffman polynomial* in the literature [9]. Here the Kauffman polynomial of a knot or link is the unique two-variable rational function

in α, s that satisfies the following Kauffman skein relations:

(i) $\times$ $-$ $\times$ $=$ $(s - s^{-1})\left(\begin{array}{c} \\ \end{array} \right)\left(- \smile \right)$,

(ii) $\curlywedge$ $=$ α $/$,

(iii) $L \sqcup \bigcirc$ $=$ δ L ,

where $\delta = \left(\dfrac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1 \right)$. We denote the Kauffman polynomial of a link L by $< L >$. Relation (iii) follows from the first two when L is nonempty. Relations (i) and (ii) are local relations, except where shown, diagrams in each are identical. A trivial closed curve in (iii) is a curve which contains no crossing and is null-homotopic. We normalize the Kauffman polynomial of the empty link $\emptyset$ to be 1.

In section 2, we study the Kauffman skein space of the 3-ball B^3 with possible boundary points. In section 3, we define linear skein maps on the Kauffman skein space of the 3-ball B^3 with four boundary points and compute the matrices of these linear maps. In section 4, we present our main theorem of calculating the Kauffman polynomial of a 2-bridge knot by decomposing it as compositions of linear skein maps from section 3. In section 5, we calculate the Kauffman polynomial of the Whitehead link as an example.

2. The Kauffman skein space of the 3-ball B^3

2.1. **The Kauffman skein space of the 3-ball B^3.** The Kauffman skein space [2] [13] of the 3-ball B^3, denoted by $K(B^3)$, is the $\mathbb{Q}(\alpha, s)$-space freely generated by framed isotopic links L in B^3 quotient by the subspace generated by the Kauffman skein relations. Given any link L in B^3, it can be simplified to $< L > \emptyset$ by applying the Kauffman skein relation, where $< L >$ is the Kauffman polynomial of L. Hence the Kauffman skein space $K(B^3)$ is generated by the empty link $\emptyset$.

2.2. **The Kauffman skein space of B^3 with four boundary points.** We place a distinguished set of four coplanar points $\{N, E, S, W\}$ on the sphere S^2, the boundary of the 3-ball B^3. A link in $(B^3, NESW)$ is a collection of closed curves and arcs joining the distinguished boundary points N, E, S, W. Two links are equivalent if one can be obtained from the other by isotopy. We define the Kauffman skein space $K(B^3, NESW)$ to be the $\mathbb{Q}(\alpha, s)$-space freely generated by framed links L in (B^3, S^2) such that $L \cap S^2 = \partial L = \{N, E, S,W\}$, considered up to an ambient isotopy fixing S^2, quotient by the subspace generated by the Kauffman skein relations. A skein element

in $K(B^3, NESW)$ is illustrated below.
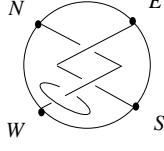
There are two natural multilinear multiplication operations in $K(B^3, NESW)$:

(1) **Concatenation.** By stacking the first on top of the second through gluing points W, S in the first with N, E in the second,

(2) **Juxtaposition.** By putting two skein elements next to each other through gluing points E, S in the first with N, W in the second,

Note that the skein element $)($ is the identity with respect to the $\odot$ operation, and the skein element $\asymp$ is the identity with respect to the $\otimes$ operation.

The Kauffman skein space $K(B^3, NESW)$ is 3-dimensional and has a basis $\{e_1, e_2, e_3\}$ [2] given by

$$e_1 = \frac{1}{s + s^{-1}} \left(s^{-1} \;)(\;\; + \;\; \times \;\; - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1}) \; \asymp \right);$$

$$e_2 = \frac{1}{s + s^{-1}} \left(s \;)(\;\; - \;\; \times \;\; + (-\delta^{-1}s + \delta^{-1}\alpha^{-1}) \; \asymp \right);$$

$$e_3 = \delta^{-1} \; \asymp .$$

In the remaining part of this section, we study properties of these basis elements which are crucial in constructing our calculation techniques.

Proposition 1. *With respect to the $\odot$ operation,*

(1) *the basis elements e_1, e_2, e_3 are orthogonal, i.e.,*

$$e_1 \odot e_2 = e_2 \odot e_1 = 0, \quad e_1 \odot e_3 = e_3 \odot e_1 = 0, \quad e_2 \odot e_3 = e_3 \odot e_2 = 0;$$

(2) *the basis elements e_1, e_2 and e_3 are idempotents, i.e., $e_1 \odot e_1 = e_1$, $e_2 \odot e_2 = e_2$, $e_3 \odot e_3 = e_3$;*

(3) *the basis elements e_1, e_2, e_3 add to the identity with respect to the $\odot$ operation, i.e., $e_1 + e_2 + e_3 = $ ⟩⟨ ;*

(4) *let $\sigma = $ ⤬ , then $\sigma \odot e_1 = e_1 \odot \sigma = se_1$, $\sigma \odot e_2 = e_2 \odot \sigma = -s^{-1}e_2$, $\sigma \odot e_3 = e_3 \odot \sigma = \alpha^{-1}e_3$. It follows that $\sigma^{-1} \odot e_1 = e_1 \odot \sigma^{-1} = s^{-1}e_1$, $\sigma^{-1} \odot e_2 = e_2 \odot \sigma^{-1} = -se_2$, $\sigma^{-1} \odot e_3 = e_3 \odot \sigma^{-1} = \alpha e_3$.*

Let $\sigma_\odot^n$ represent n copies of σ multiplied through the "$\odot$" multiplication structure, it follows that $\sigma_\odot^n \odot e_1 = e_1 \odot \sigma_\odot^n = s^n e_1$, $\sigma_\odot^n \odot e_2 = e_2 \odot \sigma_\odot^n = (-s^{-1})^n e_2$, $\sigma_\odot^n \odot e_3 = e_3 \odot \sigma_\odot^n = \alpha^{-n} e_3$.

Proof. The proofs follow by the linearity of the $\odot$ operation and (repeatedly) applying the Kauffman skein relations and substituting $\delta = \left(\dfrac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1 \right)$. Here we show $e_1 \odot e_2 = 0$ as an example.

$$e_1 \odot e_2 = \frac{1}{(s+s^{-1})^2}\left(s^{-1}\right) \Big(\, + ⤬ - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1}) ⌣⌢ \Big) \odot \left(s\right) \Big(\, - ⤬ + (-\delta^{-1}s + \delta^{-1}\alpha^{-1}) ⌣⌢ \Big)$$

$$= \frac{1}{(s+s^{-1})^2} \Big(\,\Big) \Big(\, - s^{-1} ⤬ + s^{-1}(-\delta^{-1}s + \delta^{-1}\alpha^{-1}) ⌣⌢ + s ⤬ - ⤬ \odot ⤬ + (-\delta^{-1}s + \delta^{-1}\alpha^{-1})\alpha^{-1} ⌣⌢ - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})s ⌣⌢ + (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})\alpha^{-1} ⌣⌢ - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})(-\delta^{-1}s + \delta^{-1}\alpha^{-1})\delta\; ⌣⌢ \Big) = 0. \qquad \square$$

If we rotate the basis elements e_1, e_2, e_3 in the plane by $90°$, we obtain another basis for $K(B^3, NESW)$. We present the basis elements e_1, e_2, e_3 using subscripts h (vs v) to indicate the basis elements after (vs before) the rotation:

$$e_{1v} = e_1 = \frac{1}{s+s^{-1}}\left(s^{-1}\right)\Big(\, + ⤬ - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})\; ⌣⌢ \Big);$$

$$e_{2v} = e_2 = \frac{1}{s+s^{-1}}\left(s\right)\Big(\, - ⤬ + (-\delta^{-1}s + \delta^{-1}\alpha^{-1})\; ⌣⌢ \Big);$$

$$e_{3v} = e_3 = \delta^{-1}\; ⌣⌢ \,.$$

$$e_{1h} = \frac{1}{s+s^{-1}}\left(s^{-1}\; ⌣⌢ + ⤬ - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})\; ⟩⟨ \right);$$

$$e_{2h} = \frac{1}{s+s^{-1}}\left(s\; ⌣⌢ - ⤬ + (-\delta^{-1}s + \delta^{-1}\alpha^{-1})\; ⟩⟨ \right);$$

$$e_{3h} = \delta^{-1}\; ⟩⟨ \,.$$

With respect to the $\otimes$ operation and the basis elements e_{1h}, e_{2h}, e_{3h}, similar properties of the basis elements given in Proposition 1 still hold, which we state as a corollary below.

Corollary 1.   (1) $e_{1h} \otimes e_{2h} = e_{2h} \otimes e_{1h} = 0, \quad e_{1h} \otimes e_{3h} = e_{3h} \otimes e_{1h} = 0,$
          $e_{2h} \otimes e_{3h} = e_{3h} \otimes e_{2h} = 0;$
  (2) $e_{1h} \otimes e_{1h} = e_{1h}, \; e_{2h} \otimes e_{2h} = e_{2h}, \; e_{3h} \otimes e_{3h} = e_{3h};$
  (3) $\smile\!\!\!\frown = e_{1h} + e_{2h} + e_{3h};$
  (4) let $\sigma_h = \diagdown$, then $\sigma_h \otimes e_{1h} = e_{1h} \otimes \sigma_h = s e_{1h}, \; \sigma_h \otimes e_{2h} = e_{2h} \otimes \sigma_h = -s^{-1}e_{2h},$
      $\sigma_h \otimes e_{3h} = e_{3h} \otimes \sigma_h = \alpha^{-1}e_{3h}.$

It follows that $\sigma_{h\otimes}^n \otimes e_{1h} = e_{1h} \otimes \sigma_{h\otimes}^n = s^n e_{1h}, \; \sigma_{h\otimes}^n \otimes e_{2h} = e_{2h} \otimes \sigma_{h\otimes}^n = (-s^{-1})^n e_{2h},$ $\sigma_{h\otimes}^n \otimes e_{3h} = e_{3h} \otimes \sigma_{h\otimes}^n = \alpha^{-n} e_{3h},$ where $\sigma_{h\otimes}^n$ represents n copies of σ_h multiplied through the "$\otimes$" operation.

The following are additional properties of the basis elements $e_{1v}, e_{2v}, e_{3v}, e_{1h}, e_{2h}, e_{3h}$ with respect to the $\otimes$ operation.

Proposition 2.   (1) $e_{1h} \otimes e_{1v} = e_{1v} \otimes e_{1h} = \dfrac{1}{s + s^{-1}}(s^{-1} - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{1h};$

  (2) $e_{1h} \otimes e_{2v} = e_{2v} \otimes e_{1h} = \dfrac{1}{s + s^{-1}}(-s^{-1} - \delta^{-1}s + \delta^{-1}\alpha^{-1})e_{1h};$
  (3) $e_{1h} \otimes e_{3v} = e_{3v} \otimes e_{1h} = \delta^{-1}e_{1h};$
  (4) $e_{2h} \otimes e_{1v} = e_{1v} \otimes e_{2h} = \dfrac{1}{s + s^{-1}}(-s - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{2h};$
  (5) $e_{2h} \otimes e_{2v} = e_{2v} \otimes e_{2h} = \dfrac{1}{s + s^{-1}}(s - \delta^{-1}s + \delta^{-1}\alpha^{-1})e_{2h};$
  (6) $e_{2h} \otimes e_{3v} = e_{3v} \otimes e_{2h} = \delta^{-1}e_{2h};$
  (7) $e_{3h} \otimes e_{1v} = e_{1v} \otimes e_{3h} = \dfrac{1}{s + s^{-1}}(s^{-1}\delta + \alpha - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{3h};$
  (8) $e_{3h} \otimes e_{2v} = e_{2v} \otimes e_{3h} = \dfrac{1}{s + s^{-1}}(s\delta - \alpha - \delta^{-1}s + \delta^{-1}\alpha^{-1})e_{3h};$
  (9) $e_{3h} \otimes e_{3v} = e_{3v} \otimes e_{3h} = \delta^{-1}e_{3h};$

Proof. Here we prove (1) as an example, (2)-(9) can be proved in a similar fashion.
$$e_{1h} \otimes e_{1v} = \frac{1}{s + s^{-1}}e_{1h} \otimes \left(s^{-1} \;)(\; + \diagdown - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1}) \smile\!\!\!\frown \right)$$
$$= \frac{1}{s + s^{-1}}\left(s^{-1}e_{1h} \otimes \;)(\; + e_{1h} \otimes \sigma_h^{-1} - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})e_{1h} \otimes \smile\!\!\!\frown \right)$$
$$= \frac{1}{s + s^{-1}}\left(0 + s^{-1}e_{1h} - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})e_{1h}\right)$$
$$= \frac{1}{s + s^{-1}}(s^{-1} - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{1h}. \qquad \qquad \square$$

Let M be the 3×3 matrix given by

$$M = \begin{pmatrix} \frac{1}{s+s^{-1}}(s^{-1} - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) & \frac{1}{s+s^{-1}}(-s^{-1} - \delta^{-1}s + \delta^{-1}\alpha^{-1}) & \delta^{-1} \\ \frac{1}{s+s^{-1}}(-s - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) & \frac{1}{s+s^{-1}}(s - \delta^{-1}s + \delta^{-1}\alpha^{-1}) & \delta^{-1} \\ \frac{1}{s+s^{-1}}(s^{-1}\delta + \alpha - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) & \frac{1}{s+s^{-1}}(s\delta - \alpha - \delta^{-1}s + \delta^{-1}\alpha^{-1}) & \delta^{-1} \end{pmatrix}$$

Remark. The entries of $M = (m_{ij})$ are the coefficients in Proposition 2 (1)-(9), where $e_{ih} \otimes e_{j_v} = m_{ij}e_{ih}$ for $1 \le i, j \le 3$. Notice that the matrix M is the base change matrix between the basis $\{e_{1h}, e_{2h}, e_{3h}\}$ and $\{e_{1v}, e_{2v}, e_{3v}\}$, i.e.,

$$(e_{1v}, e_{2v}, e_{3v}) = (e_{1h}, e_{2h}, e_{3h})M, \quad (e_{1h}, e_{2h}, e_{3h}) = (e_{1v}, e_{2v}, e_{3v})M.$$

It follows that $M^2 = I$, the 3×3 identity matrix.

Remark: If we change $\otimes$ to $\odot$ and exchange the subscripts v and h in identities (1)-(9) in Proposition 2, the identities still hold. We state these in the next corollary.

Corollary 2. (1) $e_{1v} \odot e_{1h} = e_{1h} \odot e_{1v} = \frac{1}{s+s^{-1}}(s^{-1} - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{1v};$

(2) $e_{1v} \odot e_{2h} = e_{2h} \odot e_{1v} = \frac{1}{s+s^{-1}}(-s^{-1} - \delta^{-1}s + \delta^{-1}\alpha^{-1})e_{1v};$

(3) $e_{1v} \odot e_{3h} = e_{3h} \odot e_{1v} = \delta^{-1}e_{1v};$

(4) $e_{2v} \odot e_{1h} = e_{1h} \odot e_{2v} = \frac{1}{s+s^{-1}}(-s - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{2v};$

(5) $e_{2v} \odot e_{2h} = e_{2h} \odot e_{2v} = \frac{1}{s+s^{-1}}(s - \delta^{-1}s + \delta^{-1}\alpha^{-1})e_{2v};$

(6) $e_{2v} \odot e_{3h} = e_{3h} \odot e_{2v} = \delta^{-1}e_{2v};$

(7) $e_{3v} \odot e_{1h} = e_{1h} \odot e_{3v} = \frac{1}{s+s^{-1}}(s^{-1}\delta + \alpha - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1})e_{3v};$

(8) $e_{3v} \odot e_{2h} = e_{2h} \odot e_{3v} = \frac{1}{s+s^{-1}}(s\delta - \alpha - \delta^{-1}s + \delta^{-1}\alpha^{-1})e_{3v};$

(9) $e_{3v} \odot e_{3h} = e_{3h} \odot e_{3v} = \delta^{-1}e_{3v}.$

3. Linear Skein Maps on $K(B^3, NESW)$ and their matrices

A wiring of a space F into another space F' is a choice of inclusion of F into F' and a choice of a set of fixed curves and arcs in $F' - F$. The wiring of F into F' induces a well-defined linear map from the skein space $K(F)$ to $K(F')$ [8]. In this section we'll consider four wirings of B^3 into itself, three of which induce linear skein maps $K(B^3, NESW) \to K(B^3, NESW)$, while the fourth one induces a linear skein map $K(B^3, NESW) \to K(B^3)$. Since $K(B^3, NESW)$ and $K(B^3)$ are vector spaces over $\mathbb{Q}(\alpha, s)$, these linear maps are linear transformations of vector spaces. In the

following, we choose $\{e_{1h}, e_{2h}, e_{3h}\}$ as the basis of $K(B^3, NESW)$ and represent these linear transformations by matrices with respect to this basis.

3.1. **The linear map $B_1(b_1)$ and the matrix $B_1(b_1)$.** Let b_1 be a nonzero integer, the linear map $B_1(b_1) : K(B^3, NESW) \to K(B^3, NESW)$, is induced by the following wiring, also called $B_1(b_1)$, for convenience
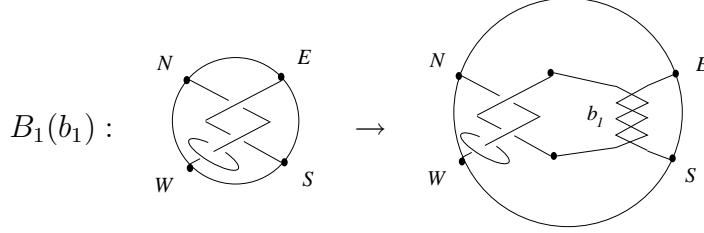
where b_1 indicates the number of crossings, it is positive if the crossings form left-hand twists, it is negative if the crossings form right-hand twists.

Lemma 1. $B_1(b_1)(xe_{1h} + ye_{2h} + ze_{3h}) = x(m_{11}s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13}\alpha^{-b_1})e_{1h} + y(m_{21}s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23}\alpha^{-b_1})e_{2h} + z(m_{31}s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33}\alpha^{-b_1})e_{3h}$.

Proof. Let $s \in K(B^3, NESW) \to K(B^3, NESW)$, then $B_1(b_1)(s) = s \otimes (\sigma_\odot^{b_1})$. Note that $\sigma_\odot^{b_1} = \sigma_\odot^{b_1} \odot \left(\, \right)\left(\right) = \sigma_\odot^{b_1} \odot (e_{1v} + e_{2v} + e_{3v}) = \sigma_\odot^{b_1} \odot e_{1v} + \sigma_\odot^{b_1} \odot e_{2v} + \sigma_\odot^{b_1} \odot e_{3v} = s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}$ from Proposition 1.

Now $B_1(b_1)(xe_{1h} + ye_{2h} + ze_{3h}) = (xe_{1h} + ye_{2h} + ze_{3h}) \otimes \sigma_\odot^{b_1}$

$= (xe_{1h} + ye_{2h} + ze_{3h}) \otimes \left(s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}\right)$

$= xe_{1h} \otimes \left(s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}\right) + ye_{2h} \otimes \left(s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}\right) + ze_{3h} \otimes \left(s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}\right)$

$= xs^{b_1}e_{1h} \otimes e_{1v} + x(-s^{-1})^{b_1}e_{1h} \otimes e_{2v} + x\alpha^{-b_1}e_{1h} \otimes e_{3v} + ys^{b_1}e_{2h} \otimes e_{1v} + y(-s^{-1})^{b_1}e_{2h} \otimes e_{2v} + y\alpha^{-b_1}e_{2h} \otimes e_{3v} + zs^{b_1}e_{3h} \otimes e_{1v} + z(-s^{-1})^{b_1}e_{3h} \otimes e_{2v} + z\alpha^{-b_1}e_{3h} \otimes e_{3v}$

$= xs^{b_1}m_{11}e_{1h} + x(-s^{-1})^{b_1}m_{12}e_{1h} + x\alpha^{-b_1}m_{13}e_{1h} + ys^{b_1}m_{21}e_{2h} + y(-s^{-1})^{b_1}m_{22}e_{2h} + y\alpha^{-b_1}m_{23}e_{2h} + zs^{b_1}m_{31}e_{3h} + z(-s^{-1})^{b_1}m_{32}e_{3h} + z\alpha^{-b_1}m_{33}e_{3h}$

$= x(m_{11}s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13})e_{1h} + y(m_{21}s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23}\alpha^{-b_1})e_{2h} + z(m_{31}s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33}\alpha^{-b_1})e_{3h}$. $\square$

We define the corresponding matrix $B_1(b_1)$ as $B_1(b_1) = (b_{ij}), 1 \le i, j \le 3$, where

$$b_{ij} = \begin{cases} m_{11}s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13}\alpha^{-b_1} & \text{if } i = j = 1 \\[2ex] m_{21}s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23}\alpha^{-b_1} & \text{if } i = j = 2 \\[2ex] m_{31}s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33}\alpha^{-b_1} & \text{if } i = j = 3 \\[2ex] 0 & \text{Otherwise.} \end{cases}$$

3.2. **The linear map $D(d_i)$ and the matrix $D(d_i)$.** Let d_i be a nonzero integer, the linear map $D(d_i) : K(B^3, NESW) \to K(B^3, NESW)$ is induced by the wiring
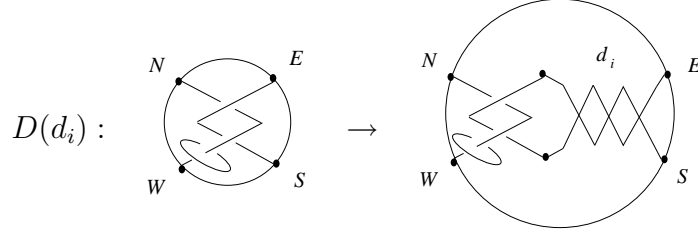
Similarly d_i indicates the number of crossings, it is positive if the crossings form left-hand twists, it is negative if the crossings form right-hand twists.

Lemma 2. $D(d_i)(xe_{1h} + ye_{2h} + ze_{3h}) = xs^{d_i}e_{1h} + y(-s^{-1})^{d_i}e_{2h} + z\alpha^{-d_i}e_{3h}$

$$= (e_{1h} \ e_{2h} \ e_{3h}) \begin{pmatrix} s^{d_i} & 0 & 0 \\ 0 & (-s^{-1})^{d_i} & 0 \\ 0 & 0 & \alpha^{-d_i} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Proof. Note that $\sigma_{h\otimes}^{d_i} = \sigma_{h\otimes}^{d_i} \otimes \left(\smile \atop \frown \right) = \sigma_{h\otimes}^{d_i} \otimes (e_{1h} + e_{2h} + e_{3h})$
$= s^{d_i}e_{1h} + (-s^{-1})^{d_i}e_{2h} + \alpha^{-d_i}e_{3h}$ by the idempotent properties of the basis elements. Now by substitution, $D(d_i)(xe_{1h} + ye_{2h} + ze_{3h}) = (xe_{1h} + ye_{2h} + ze_{3h}) \otimes (\sigma_{h\otimes}^{d_i})$
$= (xe_{1h} + ye_{2h} + ze_{3h}) \otimes (s^{d_i}e_{1h} + (-s^{-1})^{d_i}e_{2h} + \alpha^{-d_i}e_{3h})$
$= xs^{d_i}e_{1h} + y(-s^{-1})^{d_i}e_{2h} + z\alpha^{-d_i}e_{3h}.$

$\square$

We define the corresponding matrix $D(d_i)$ as

$$D(d_i) = \begin{pmatrix} s^{d_i} & 0 & 0 \\ 0 & (-s^{-1})^{d_i} & 0 \\ 0 & 0 & \alpha^{-d_i} \end{pmatrix}.$$

3.3. **The linear map $B(b_i)$ and the matrix $B(b_i)$.** Let b_i be a nonzero integer, the linear map $B(b_i) : K(B^3, NESW) \to K(B^3, NESW)$ is induced by the wiring
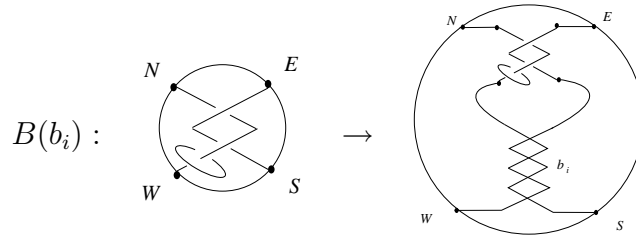
where b_i indicates the number of crossings, it is positive if the crossings form left-hand twists, it is negative if the crossings form right-hand twists.

Lemma 3. $B(b_i)(xe_{1h}+ye_{2h}+ze_{3h}) = (e_{1h}\ e_{2h}\ e_{3h})M \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$;

where M is the base change matrix between the basis $\{e_{1h}, e_{2h}, e_{3h}\}$ and $\{e_{1v}, e_{2v}, e_{3v}\}$.

Proof. Note $\sigma_{\odot}^{b_i} = \sigma_{\odot}^{b_i} \odot \ )(= \sigma_{\odot}^{b_i} \odot (e_{1v}+e_{2v}+e_{3v}) = s^{b_1}e_{1v}+(-s^{-1})^{b_1}e_{2v}+\alpha^{-b_1}e_{3v}$,

$B(b_i)(xe_{1h} + ye_{2h} + ze_{3h}) = (xe_{1h} + ye_{2h} + ze_{3h}) \odot (\sigma_{\odot}^{b_i})$

$= (xe_{1h} + ye_{2h} + ze_{3h}) \odot (s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v})$

$= (xe_{1h} + ye_{2h} + ze_{3h}) \odot (s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}))$

$= (xe_{1h}+ye_{2h}+ze_{3h}) \odot s^{b_1}e_{1v} + (xe_{1h}+ye_{2h}+ze_{3h}) \odot (-s^{-1})^{b_1}e_{2v} + (xe_{1h}+ye_{2h}+ze_{3h}) \odot \alpha^{-b_1}e_{3v}$

$= (xm_{11}+ym_{12}+zm_{13})s^{b_1}e_{1v} + (xm_{21}+ym_{22}+zm_{23})(-s^{-1})^{b_1}e_{2v} + (xm_{31}+ym_{32}+zm_{33})\alpha^{-b_1}e_{3v}$
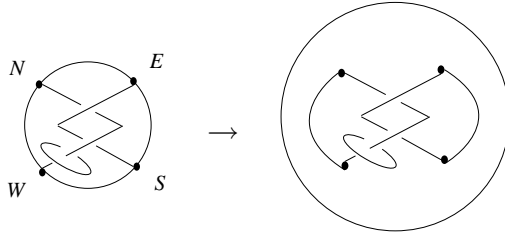
$= (e_{1v}\ e_{2v}\ e_{3v}) \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$= (e_{1h}\ e_{2h}\ e_{3h})M \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, as $(e_{1v}\ e_{2v}\ e_{3v}) = (e_{1h}\ e_{2h}\ e_{3h})M$.

$\qquad\square$

We define the corresponding matrix $B(b_i)$ as

$$B(b_i) = M \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M.$$

3.4. **The closure-map C and the matrix C.** Finally the linear map $C : K(B^3, NESW) \to K(B^3)$ is induced by the closure wiring:

Lemma 4.
$$C(xe_{1h} + ye_{2h} + ze_{3h}) = z\delta\emptyset,$$
where $\emptyset$ represents the empty link which generates $K(B^3)$.

Proof. The closure of e_{1h} is zero and the closure of e_{2h} is also zero by the orthogonal properties. The closure of e_{3h} can be simplified as $\delta^{-1}\delta^2\emptyset = \delta\emptyset$. $\qquad\square$

We therefore define the matrix $C = (0, 0, \delta)$.

4. The Kauffman Polynomials of the 2-bridge knots

The 2-bridge knot with continuous fraction notation $F(D) = [b_1, d_1, b_2, d_2, \cdots, d_n, b_{n+1}]$ is an image of the compositions of wiring maps defined in last section. We summarize our main results in:

Theorem 1. *Let $F(D) = [b_1, d_1, b_2, d_2, \cdots, d_n, b_{n+1}]$ be the 2-bridge knot given in section 1, then the Kauffman polynomial of D is*

$$< D >= (0, 0, \delta) B(b_{n+1}) D(d_n) \cdots B(b_{i+1}) D(d_i) \cdots B(b_2) D(d_1) B_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where $B_1(b_1), B(b_i), D(d_i)$ are matrices defined in the previous section as

$$B_1(b_1) = (b_{ij}), 1 \le i, j \le 3, \text{ with } b_{ij} = \begin{cases} m_{11} s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13} \alpha^{-b_1} & \text{if } i = j = 1 \\ m_{21} s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23} \alpha^{-b_1} & \text{if } i = j = 2 \\ m_{31} s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33} \alpha^{-b_1} & \text{if } i = j = 3 \\ 0 & \text{Otherwise} \end{cases}$$

$$B(b_i) = M \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M, \text{ and } D(d_i) = \begin{pmatrix} s^{d_i} & 0 & 0 \\ 0 & (-s^{-1})^{d_i} & 0 \\ 0 & 0 & \alpha^{-d_i} \end{pmatrix}.$$

Proof. Using the linear maps defined in the previous section and their compositions, the 2-bridge knot $D = C \circ B(b_{n+1}) \circ D(d_n) \circ \cdots \circ B(b_{i+1}) \circ D(d_i) \circ \cdots \circ B(b_2) \circ D(d_1) \circ B_1(b_1)(\overset{\smile}{\underset{\frown}{}})$.

As each of these maps is a linear transformation between vector spaces, it can be represented by its matrix with respect to the basis $\{e_{1h}, e_{2h}, e_{3h}\}$. Note $\overset{\smile}{\underset{\frown}{\times}} = e_{1h} + e_{2h} + e_{3h}$, so

$D = C \circ B(b_{n+1}) \circ D(d_n) \circ \cdots \circ B(b_{i+1}) \circ D(d_i) \circ \cdots \circ B(b_2) \circ D(d_1) \circ B_1(b_1)(e_{1h} + e_{2h} + e_{3h})$,

$$= C \circ B(b_{n+1}) \circ D(d_n) \circ \cdots \circ B(b_{i+1}) \circ D(d_i) \circ \cdots \circ B(b_2) \circ D(d_1) \left(\begin{pmatrix} e_{1h} & e_{2h} & e_{2h} \end{pmatrix} B_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right),$$

in matrices,

$$= C \left(\begin{pmatrix} e_{1h} & e_{2h} & e_{2h} \end{pmatrix} B(b_{n+1}) D(d_n) \cdots B(b_{i+1}) D(d_i) \cdots B(b_2) D(d_1) B_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$= (0,0,\delta\emptyset)B(b_{n+1})D(d_n)\cdots B(b_{i+1})D(d_i)\cdots B(b_2)D(d_1)B_1(b_1)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

by Lemma 4.

Take the Kauffman polynomial, we have

$$< D >= (0,0,\delta)B(b_{i+1})D(d_n)\cdots B(b_{i+1})D(d_i)\cdots B(b_2)D(d_1)B_1(b_1)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

since the Kauffman polynomial of the empty link $\emptyset$ is $< \emptyset >= 1$. $\square$

5. An Example–The Kauffman polynomial of the Whitehead link

Here we demonstrate how to calculate the Kauffman polynomial of the Whitehead link using linear maps and matrices. We choose the diagram W with the continued fraction notation $F(W) = [-2, 1, -2]$ for the Whitehead link, then the corresponding matrices are as follows,

$$B_1(-2) = (b_{ij}), 1 \le i,j \le 3, \ where \ b_{ij} = \begin{cases} m_{11}s^{-2} + m_{12}s^2 + m_{13}\alpha^2 & \text{if } i=j=1 \\ m_{21}s^{-2} + m_{22}s^2 + m_{23}\alpha^2 & \text{if } i=j=2 \\ m_{31}s^{-2} + m_{32}s^2 + m_{33}\alpha^2 & \text{if } i=j=3 \\ 0 & \text{Otherwise} \end{cases}$$

$$D(1) = \begin{pmatrix} s & 0 & 0 \\ 0 & -s^{-1} & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \text{ and } B(-2) = M\begin{pmatrix} s^{-2} & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}M,$$

According to Theorem 1, the Kauffman polynomial of W is

$$< W >= (0,0,\delta)B(-2)D(1)B_1(-2)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{1}{\alpha^2\delta s^4}(\alpha^7 s^4 - 2\alpha^4(-1+\delta^2)s^3(-1+s^2) + \alpha^3(1-s^2+s^4)(1+(-1+2\delta)s^2+s^4) - (-1+\delta)\delta s(-1+2s^2-2s^4+s^6) + \alpha^2(-2+\delta+\delta^2)s(-1+2s^2-2s^4+s^6) - 2\alpha^5(s^2-s^4+s^6) + \alpha(s^2(-1+s^2)^2 - \delta^2s^2(-1+s^2)^2 + \delta^3(s^2-s^4+s^6) + \delta(-1+s^2-2s^4+s^6-s^8)))$. \text{ Substitute}$$

in $\delta = \left(\frac{\alpha-\alpha^{-1}}{s-s^{-1}}+1\right)$, $< W >= \frac{1}{\alpha^3 s^4(-1+s^2)^2}(-\alpha^2+\alpha^4-\alpha^3 s+\alpha^5 s+s^2+\alpha^2 s^2 - \alpha^4 s^2 - \alpha^6 s^2 + \alpha s^3 + 2\alpha^3 s^3 - 2\alpha^5 s^3 - \alpha^7 s^3 - 2s^4 + \alpha^4 s^4 + 2\alpha^6 s^4 - \alpha s^5 - 3\alpha^3 s^5 + \alpha^5 s^5 + 3\alpha^7 s^5 + 3s^6 - 2\alpha^2 s^6 - \alpha^4 s^6 - 2\alpha^6 s^6 + \alpha s^7 + 3\alpha^3 s^7 - \alpha^5 s^7 - 3\alpha^7 s^7 - 2s^8 + \alpha^4 s^8 + 2\alpha^6 s^8 - \alpha s^9 - 2\alpha^3 s^9 + 2\alpha^5 s^9 + \alpha^7 s^9 + s^{10} + \alpha^2 s^{10} - \alpha^4 s^{10} - \alpha^6 s^{10} + \alpha^3 s^{11} - \alpha^5 s^{11} - \alpha^2 s^{12} + \alpha^4 s^{12})$.

Our calculations are carried out using Mathematica.

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